The Quantum EM Field as a Stochastic Process

PHYS 882 final project

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Spatial measurements of the vacuum electric field may be understood in terms of stochastic processes, particularly Gaussian white noise.

INTRODUCTION

Different types of measurables correspond to different types of random variables. The number operator corresponds to random natural numbers with one probability per number; position and momentum operators correspond to random real numbers with probability density functions. What kind of random variable does a quantum field correspond to? The classical object to which it corresponds is not one real number but an entire function. Therefore a quantum field corresponds to a random function, otherwise known as a stochastic process. Thus stochastic processes are a natural part of the statistical properties of quantum fields. This project examines the role of stochastic processes in quantum mechanics. Measurements of the electric field in a vacuum will be examined particularly closely.

GAUSSIAN WHITE NOISE

Denote Gaussian white noise $N(x)$. How should it be defined? It is meant to be a stochastic process with a Gaussian distribution and zero spatial correlation. It is easy enough to guess at a good model for such a process. The obvious model is to map mutually independent standard normal random variables to every point on the real line. The problem comes when you try to integrate over the resulting stochastic process. Start with a Riemann sum:

$$\int_{x=0}^{L} N(x) = \lim_{N\to\infty} \frac{L}{N} \sum_{i=0}^{N-1} G_i,$$

where $\{G_i\}$ are mutually independent standard normal random variables. Using the general rule for the addition of independent Gaussian random variables, the above Riemann sum becomes

$$\int_{x=0}^{L} N(x) = \lim_{N\to\infty} \frac{L}{N}\sqrt{N}\, G = 0$$

Thus any attempt at integrating the naïve Gaussian white noise over a finite interval results in zero.

Since we need to be able to integrate over our stochastic process, a different definition is needed. Define Gaussian white noise as a stochastic process N with the following two properties. Firstly, the integral of the process over an interval of length h results in a normal distribution with standard deviation $\sqrt{h}$.

$$\int_{x=a}^{a+h} N(x) = \sqrt{h}\, G, \tag{1}$$

where G is a random variable with a standard normal distribution. Secondly, integrals of N over non-overlapping sets are mutually independent. That is if you define a set of N random variables as follows:

$$Y_i = \int_{x\in D_i} N(x), \tag{2}$$

where $\{D_i\}$ are mutually disjoint subsets of the real line and N is a positive integer, then the set of random variables $\{Y_i\}$ is mutually independent by definition.

One immediate consequence of this definition is that the standard deviation of the process's average over a given window in space approaches infinity as the window width approaches zero. So although the process can be integrated, as a consequence, it must be integrated in order to be well-defined, because the process's value at a specific point in space is infinite.

This process is useful for constructing other processes. For instance, the Wiener process, used to describe Brownian motion and other space- and time-continuous random walks is just the integral of Gaussian white noise:

$$w(x) = \int_{u=0}^{x} N(u) \tag{3}$$

GAUSSIAN WHITE NOISE'S SPECTRAL POWER DENSITY

The convention for the Fourier transforms that will be used throughout is

$$\tilde{f}(k) = \mathcal{F}f(k) = \frac{1}{\sqrt{\tau}} \int_x e^{-ikx} f(x). \tag{4}$$

Using this convention, the common theorems surrounding the Fourier transform have the following form:

$$f(x) = \mathcal{F}^{-1}\tilde{f}(x) = \frac{1}{\sqrt{\tau}}\int_k e^{ikx}\tilde{f}(k) \tag{5}$$

$$\mathcal{F}(-i\mathrm{D}f)(k) = k\tilde{f}(k) \tag{6}$$

$$\mathcal{F}(fg) = \frac{1}{\sqrt{\tau}}\tilde{f} * \tilde{g} \tag{7}$$

$$\int_x f(x)^* g(x) = \int_k \tilde{f}(k)^* \tilde{g}(k) \tag{8}$$

The power spectral density (power per unit wavenumber) of a stationary stochastic process F can be determined using a property of the process called its *autocorrelation*. F's autocorrelation is defined

$$r_{FF}(h) = \langle F(x)F(x-h)\rangle\,. \tag{9}$$

This function does not depend on x because the stochastic process is stationary, i.e. its behaviour is the same over all space. Substituting white noise into this formula won't be immediately possible, since white noise cannot be evaluated at a point in space. Instead, the following integral will be examined:

$$\begin{aligned} I &= \int_{xy} f(x)\,\langle N(x)N(y)\rangle\, g(y) \\ &= \left\langle \int_x f(x)N(x) \int_y g(y)N(y) \right\rangle \end{aligned}$$

This can be approximated by dividing the whole real line into equal segments of length h and considering f and g constant in each section.

$$\begin{aligned} I &= \sum_{mn} \left\langle \int_{x=x_m}^{x_m+h} f(x)N(x) \int_{y=y_m}^{y_m+h} g(y)N(y) \right\rangle \\ &\approx \sum_{mn} f(x_m)g(x_n) \left\langle \int_{x=x_m}^{x_m+h} N(x) \int_{y=x_m}^{x_m+h} N(y) \right\rangle \end{aligned}$$

Applying the definition of Gaussian white noise,

$$I \approx h\sum_{mn} f(x_m)g(x_n)\,\langle G_m G_n\rangle\,,$$

where G_n are mutually independent. Since G_n^2 is a χ_1^2 distribution, its expected value is 1. All of the other $\langle G_m G_n\rangle$ are zero, since they are independent and G_n's mean is zero. As a result,

$$I \approx h\sum_n f(x_n)g(x_n).$$

As $h \to 0$, this results in an integral over x, giving the expression

$$\int_{xy} f(x)\,\langle N(x)N(y)\rangle\, g(y) = \int_x f(x)g(x).$$

This is true for all f and g. If translated delta functions are substituted in their place ($f(x) = \delta(x-z)$, $g(y) = \delta(y-w)$), this results in the equation

$$\langle N(x)N(y)\rangle = \delta(y-x) \tag{10}$$

Therefore the autocorrelation of Gaussian white noise is

$$r_{NN}(h) = \delta(h) \tag{11}$$

Now, we can find the spectral power density of white noise. As was mentioned earlier, the autocorrelation can be used to determine this property. The exact formula is

$$\rho_k(k) = \frac{1}{\sqrt{\tau}}\tilde{r}_{FF}(k) \tag{12}$$

This result is the Wiener-Khinchin theorem[1] and it is magnificent. Substituting in r_{NN},

$$\rho_k(k) = \frac{1}{\sqrt{\tau}}\mathcal{F}\delta \tag{13}$$

$$\rho_k(k) = \frac{1}{\tau} \tag{14}$$

Thus the spectral power density is completely flat, extending out to $k = \pm\infty$, just as one would expect of white noise.

QUADRATURE REPRESENTATION OF SINGLE MODE STATES

In this course, single mode states were represented with the number state basis. But single mode states can also be expressed using the eigenstates of the quadrature operator, $Q = \frac{1}{\sqrt{2}}(a + a^\dagger)$, to which the mode's E field is proportional. Q's eigenstates form a continuous basis over all reals. This basis can be used to determine the probability density of measurements of the Q operator, thus determining the probability density of the E field for a given state. These results for single mode states will be needed to determine the general statistical properties of the whole E field as a stochastic process. This section will derive and prove the results that will be required for this.

We'll start with some definitions. Define a ladder operator a and vacuum state $|0\rangle$ by these properties:

$$[a, a^\dagger] = 1 \tag{15}$$

$$a\,|0\rangle = 0 \tag{16}$$

Define the quadrature operators and the number states in terms of the ladder operator and the vacuum state:

$$Q = \frac{1}{\sqrt{2}}(a + a^\dagger) \tag{17}$$

$$P = \frac{-i}{\sqrt{2}}(a - a^\dagger) \tag{18}$$

$$|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger n}\,|0\rangle \tag{19}$$

By this convention,

$$N + \tfrac{1}{2} = \tfrac{1}{2}(P^2 + Q^2), \text{ and} \tag{20}$$

$$-i[Q, P] = 1. \tag{21}$$

The easiest way to find Q's eigenvectors is to construct the equivalent 1D quantum harmonic oscillator system:

$$H = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^2\right) \tag{22}$$

$$H f_n = \left(n + \tfrac{1}{2}\right) f_n \tag{23}$$

$$\int_q f_m f_n = \delta_{mn} \tag{24}$$

The energy representation of the position operator, $\int_q f_m q f_n$, is identical to the number representation of the Q quadrature operator, $\langle m| Q |n\rangle$. Therefore $f_n(q)$ can be used to construct the eigenvectors of Q.

$$\begin{aligned} Q \sum_n |n\rangle f_n(q) &= \sum_{mn} |m\rangle\langle m| Q |n\rangle f_n(q) \\ &= \sum_{mn} |m\rangle \int_{q'} f_m(q') q' f_n(q') f_n(q) \\ &= \sum_{m} |m\rangle \int_{q'} f_m(q') q' \delta(q - q') \\ &= q \sum_m |m\rangle f_m(q) \end{aligned}$$

Therefore Q's eigenvalues span the entire real line and its eigenvectors, in terms of the already-defined number basis, are

$$|q\rangle = \sum_n |n\rangle f_n(q). \tag{25}$$

This basis can be used to determine the probability distribution of Q measurements. For instance, the number states have the probability distribution

$$\rho_n(q) = |\langle q|n\rangle|^2 = f_n(q)^2 \tag{26}$$

In particular, the vacuum state has a standard normal distribution with a standard deviation of $\frac{1}{\sqrt{2}}$.

The Q representations of the quadrature operators are

$$Q\psi(q) = q\psi(q) \tag{27}$$

$$P\psi(q) = -i\frac{\mathrm{d}}{\mathrm{d}q}\psi(q) \tag{28}$$

A CONTINUOUS WAVENUMBER SETUP

The physical setup that will be used to demonstrate the measurement of the quantum EM field as a stochastic process will be a nondispersive waveguide of infinite length and effective cross-sectional area, S. Use the Coulomb gauge. Define representative scalar values of $E(x)$ and $A(x)$ at each point along the waveguide such that the classical expression for the field energy inside the waveguide is

$$H_{cl} = \frac{\varepsilon_0 S}{2} \int_x \left(E^2 + c^2 \frac{\partial A}{\partial x}^2\right) \tag{29}$$

and the field equations are

$$\dot{A} = -E \tag{30}$$

$$\dot{E} = \frac{\partial^2 A}{\partial x^2}. \tag{31}$$

This quantum system will be represented with the Hamiltonian

$$H = \int_k \hbar c\,|k|\,\tilde{a}^\dagger \tilde{a} \tag{32}$$

Unlike the multimode Hamiltonians covered in the course, which had quantised wavenumber, the ladder operator in this Hamiltonian is a continuous ladder operator in k. This comes as a result of the fact that the volume of the waveguide is infinite. a has the defining properties

$$[a(x), a(y)^\dagger] = \delta(y - x) \tag{33}$$

$$a(x)\,|0\rangle = 0 \tag{34}$$

The Fourier transform of a is also a continuous ladder operator with the same properties.

The Fourier transformed electric field and vector potential operators, in terms of a, are:

$$\tilde{E}(k) = \sqrt{\frac{\hbar c\,|k|}{2\varepsilon_0 S}}(\tilde{a}(k) + \tilde{a}(-k)^\dagger) \tag{35}$$

$$\tilde{A}(k) = -i\sqrt{\frac{\hbar}{2\varepsilon_0 S c\,|k|}}(\tilde{a}(k) - \tilde{a}(-k)^\dagger) \tag{36}$$

It is trivial to ensure the Heisenburg versions of E and A follow the classical field equations.

The a operator can't be used directly to create particles. It has to be combined with normalised mode functions. In the following equation, b is a regular ladder operator, annihilating particles in the mode specified by f.

$$b = \int_x f(x)^* a(x), \text{ where } \int_x |f(x)|^2 = 1 \tag{37}$$

It is easily proven that b satisfies the requirements of a regular ladder operator, namely $[b, b^\dagger] = 1$ and $b\,|0\rangle = 0$. As a regular ladder operator, quadrature measurables and single mode states for mode f can be constructed from from it. These single mode states could be number states or any other state accessible from the basis of number states.

THE VACUUM STATE AS A STOCHASTIC PROCESS

We'll now construct and examine the simplest space-dependent field measurable possible: $\phi(x) = \frac{1}{\sqrt{2}}(a(x) + a^\dagger(x))$. Just a would be simpler, but it cannot be measured, since it is not Hermitian. We'll now apply this measurable to the vacuum state and completely characterise this measurement as a stochastic process.

Since $\phi(x)$ is not just one measurable, but infinitely many measurables, before it can be measured as a whole, it must be confirmed that all of these measurables can be measured simultaneously, which requires that they all commute. Proving this is trivial enough not to bother including, but it is an important point nonetheless. For instance, it is the reason why the E and B fields cannot be measured simultaneously.

ϕ can't be measured exactly at a point in space for reasons that will soon become clear. Its integral over an interval h can be measured, however. Define the measurable

$$M = \int_{x=x_0}^{x_0+h} \phi(x). \tag{38}$$

M's statistical properties can be quantified using the regular ladder operator $b = \int_x W(x)a(x)$ for the mode specified by the normalised window function

$$W(x) = \begin{cases} \frac{1}{\sqrt{h}} & x \in [x_0, x_0 + h] \\ 0 \end{cases} \tag{39}$$

Substituting this into M,

$$\begin{aligned} M &= \sqrt{h} \int_x W(x)\phi(x) \\ &= \sqrt{\frac{h}{2}} \int_x W(x)(a(x) + a^\dagger(x)) \\ &= \sqrt{\frac{h}{2}}(b + b^\dagger) \\ &= \sqrt{h}Q \end{aligned}$$

Thus M has statistical properties identical to the Q-quadrature operator for mode W, except that it is scaled by $\sqrt{h}$. Since from the section on the quadrature representation, Q has a Gaussian distribution with mean 0 and standard deviation $\frac{1}{\sqrt{2}}$ for the vacuum state, M is also Gaussian for the vacuum state, but with mean 0, standard deviation $\sqrt{\frac{h}{2}}$.

So, when measuring the vacuum, the integral of ϕ over an interval of size h produces a Gaussian distribution of measurement results with a mean of 0 and a standard deviation proportional to the square root of the interval size.

$$\int_{x=x_0}^{x_0+h} \phi(x) = \sqrt{\frac{h}{2}} G \quad \text{(for vacuum)} \tag{40}$$

This strongly suggests that ϕ is a Gaussian white noise process for this measurement. This is indeed the case, but mutual independence of non-overlapping integrals must still be proven to satisfy all criteria of white noise.

To test for mutual independence, n modes derived from non-overlapping normalised square windows like W will be created with varying window width h_a. Quadrature measurables, Q_a, will then be created and simultaneously measured, resulting in a random vector $Q \in \Omega \to \mathbb{R}^n$. This vector's probability density function $\rho(q)$, where $q \in \mathbb{R}^n$, will then be determined. If ρ can be factored into a product of single-input functions $\prod_a f_a(q_a)$, then Q_a are mutually independent.

Start with the ranges. Define non-overlapping ranges $D_a \subset \mathbb{R}$ and denote the width of D_a with h_a. Define a normalised window function for each range.

$$W_a(x) = \begin{cases} \frac{1}{\sqrt{h_a}} & x \in D_a \\ 0 \end{cases} \tag{41}$$

Define a mode for each range with ladder operator b_a.

$$b_a = \int_x W_a(x)a(x) \tag{42}$$

To make the coming math easier, define an operator $c_a(q)$ that creates a quadrature eigenstate in mode a.

$$c_a(q) = \sum_{n=0}^{\infty} \frac{f_n(q)}{\sqrt{n!}} b_a^{\dagger n}, \tag{43}$$

where f_n are the quadrature representations of the number states, as defined in the section on the quadrature representation. Use this operator to define the eigenstates of the vector measurable Q.

$$|q\rangle = \prod_a c_a(q_a)\,|0\rangle \tag{44}$$

It must still be proven that $|q\rangle$ are the eigenstates of Q. This will require rearrangement of the factors of the above product, which requires that the ladder operators all commute with each other. Here is the proof:

$$\begin{aligned} [b_a, b_b] &= \int_{xy} W_a(x)[a(x), a(y)]W_b(y) \\ &= 0 \\ [b_a, {b_b}^\dagger] &= \int_{xy} W_a(x)[a(x), a(y)^\dagger]W_b(y) \\ &= \int_{xy} W_a(x)\delta(y - x)W_b(y) \\ &= \int_x W_a(x)W_b(x) \\ &= \delta_{ab} \end{aligned}$$

Using this result,

$$\begin{aligned} Q_a \left| q \right\rangle &= Q_a \prod_b c_b(q_b) \left| 0 \right\rangle \\ &= \prod_{b|b\neq a} c_b(q_b) \;\; Q_a c_a(q_a) \left| 0 \right\rangle \\ &= \prod_{b|b\neq a} c_b(q_b) \;\; q_a c_a(q_a) \left| 0 \right\rangle \\ &= q_a \left| q \right\rangle \end{aligned}$$

Therefore $Q \left| q \right\rangle = q \left| q \right\rangle$. With this confirmed, $\left| q \right\rangle$ may now be used to determine the probability distribution function of Q.

$$\begin{aligned} \rho(q) &= \left| \langle 0 | q \rangle \right|^2 \\ &= \left| \langle 0 | \prod_a c_a(q_a) \left| 0 \right\rangle \right|^2 \end{aligned}$$

All terms of c_a are zeroed by $\langle 0 |$ except for term zero, leaving

$$\begin{aligned} \rho(q) &= \left| \langle 0 | \prod_a f_0(q_a) \left| 0 \right\rangle \right|^2 \\ &= \prod_a f_0(q_a)^2 \end{aligned}$$

The probability density function factors into a product, exactly as required. Therefore non-overlapping integrals of ϕ are mutually independent and the ϕ operator is a Gaussian white noise process when measuring the vacuum state:

$$\phi_v = \frac{1}{\sqrt{2}} N \quad \text{(for vacuum)} \tag{45}$$

E FIELD STATISTICS IN VACUUM

ϕ's Fourier transform is

$$\tilde{\phi}(x) = \frac{1}{\sqrt{2}} (\tilde{a}(k) + \tilde{a}(-k)^\dagger) \tag{46}$$

Therefore in the Fourier domain, $\tilde{E}$ is just a filtered version of $\tilde{\phi}$.

$$\tilde{E} = \tilde{Z}\tilde{\phi}, \text{ where} \tag{47}$$

$$\tilde{Z}(k) = \sqrt{\frac{\hbar c \left| k \right|}{\varepsilon_0 S}} \tag{48}$$

That makes E a convolution of ϕ.

$$E = \frac{1}{\sqrt{\tau}} Z * \phi \tag{49}$$

Thus ϕ can be used to determine the statistical properties of the E field in a vacuum. From equation 45,

$$E_v = \frac{1}{\sqrt{2\tau}} Z * N. \tag{50}$$

Let's measure average values of E with a generic averaging kernel W.

$$W(x) = \tfrac{1}{h} W_0(\tfrac{x}{h}), \text{ where} \tag{51}$$

$$\int_u W_0(u) = 1 \tag{52}$$

The average can be performed with the inner product

$$\langle f, g \rangle = \int_x f(x)^* g(x). \tag{53}$$

Using this inner product, the average E field is

$$\overline{E}_v = \frac{1}{\sqrt{2\tau}} \langle W, Z * N \rangle \tag{54}$$

Convolutions with symmetric convolution kernels are Hermitian under this inner product. Therefore

$$\overline{E}_v = \frac{1}{\sqrt{2\tau}} \langle Z * W, N \rangle . \tag{55}$$

In general, if $f(x)$ is real, then

$$\langle f, N \rangle = \| f \| \, G, \tag{56}$$

where G is standard normal and $\| f \| = \sqrt{\langle f, f \rangle}$. To prove this, subdivide $\mathbb{R}$ into ranges of width ε: $D_a = [x_a, x_a + \varepsilon]$, where $x_a = a\varepsilon$ and $a \in \mathbb{Z}$.

$$\begin{aligned} \langle f, N \rangle &= \sum_a \int_{x \in D_a} f(x) N(x) \\ &= \lim_{\varepsilon \to 0} \sum_a f(x_a) \int_{x \in D_a} N(x) \\ &= \lim_{\varepsilon \to 0} \sum_a f(x_a) \sqrt{\varepsilon}\, G_a \quad \text{(from Eqn. 1)} \end{aligned}$$

Since G_a are independent, the rule for the addition of Gaussians ($\sigma G + \pi H = \sqrt{\sigma^2 + \pi^2}\, J$) can be used repeatedly to yield

$$\begin{aligned} \langle f, N \rangle &= \lim_{\varepsilon \to 0} \sqrt{\varepsilon \sum_a f(x_a)^2}\, G \\ &= \sqrt{\int_x f(x)^2}\, G \\ &= \| f \| \, G \quad \text{Q.E.D.} \end{aligned}$$

This result can be used to determine the distribution of vacuum E field measurements.

$$\begin{aligned}
\overline{E}_v &= \frac{1}{\sqrt{2\tau}} \|Z * W\| G \\
&= \frac{1}{\sqrt{2}} \left\|\tilde{Z}\tilde{W}\right\| G \\
&= \sqrt{\frac{\hbar c}{2\varepsilon_0 S} \int_k |k|\, |\tilde{W}|^2}\; G \quad \text{(from Eqn. 48)} \\
&= \sqrt{\frac{\hbar c}{2\varepsilon_0 S} \int_k |k|\, |\tilde{W}_0(hk)|^2}\; G \quad \text{(from Eqn. 51)} \\
\overline{E}_v &= \frac{1}{h}\sqrt{\frac{\hbar c}{2\varepsilon_0 S} \int_u |u|\, |\tilde{W}_0(u)|^2}\; G
\end{aligned}$$

Thus $\overline{E}_v$ has a Gaussian distribution with zero mean and standard deviation

$$\sigma(\overline{E}_v) = \frac{1}{h}\sqrt{\frac{\hbar c}{2\varepsilon_0 S} \int_u |u|\, |\tilde{W}_0(u)|^2}. \tag{57}$$

This is a very interesting result. First of all, there's the h dependence. For white noise, the average is inversely proportional to the square root of the window width, but the average of the vacuum E field is inversely proportional to just the window width. Thus as the window shrinks, the average diverges even faster than for white noise.

The second, perhaps even stranger implication comes from the $|u|$ factor inside the integral. This causes the integral to diverge for many windows, including some seemingly innocuous windows like the square window. Thus it is impossible to measure the vacuum E field with a perfectly square averaging window!

Now, let's examine the covariance of two simultaneous vacuum E measurements. Use windows W_0 and W_1 for the two measurements.

$$\overline{E}_a = \langle W_a, E_v \rangle \text{, where} \tag{58}$$

$$\int_x W_a = 1 \tag{59}$$

The covariance is

$$\begin{aligned}
\left\langle \overline{E}_0 \overline{E}_1 \right\rangle &= \frac{1}{2\tau} \langle \langle Z * W_0, N\rangle \langle Z * W_1, N\rangle\rangle \quad \text{(from Eqn. 55)} \\
&= \frac{1}{2\tau} \langle Z * W_0, Z * W_1\rangle \quad \text{(from Eqn. 10)} \\
&= \frac{1}{2\tau} \langle W_0, Z * Z * W_1\rangle \\
&= \frac{1}{2\sqrt{\tau}} \langle W_0, Y * W_1\rangle ,
\end{aligned}$$

where

$$\tilde{Y} = |\tilde{Z}|^2 = \frac{\hbar c\, |k|}{\varepsilon_0 S} \quad \text{(from Eqn. 48)} \tag{60}$$

Suppose the windows are well-localised at x_0 and x_1 respectively with zero overlap. Then the following approximation can be used:

$$\begin{aligned}
\left\langle \overline{E}_0 \overline{E}_1 \right\rangle &\approx \frac{1}{2\sqrt{\tau}} Y(x_1 - x_0) \int_x W_0 \int_x W_1 \\
\left\langle \overline{E}_0 \overline{E}_1 \right\rangle &\approx \frac{1}{2\sqrt{\tau}} Y(x_1 - x_0)
\end{aligned}$$

Doing the inverse Fourier transform on Y, we get

$$Y(x) = -\frac{\hbar c}{\tau \varepsilon_0 S x^2} \tag{61}$$

Substituting this into the covariance,

$$\left\langle \overline{E}_0 \overline{E}_1 \right\rangle \approx -\frac{\hbar c}{\tau \varepsilon_0 S (x_1 - x_0)^2} \tag{62}$$

Thus measurements of the vacuum E field are actually negatively correlated—an effect with an inverse-squared dropoff.

Now, let's find the vacuum E field's power spectral density. First, we'll need its autocorrelation.

$$\begin{aligned}
r_{EE}(h) &= \langle E_v(x) E_v(x - h)\rangle \\
&= \frac{1}{2\tau} \langle (Z * N)(x)(Z * N)(x - h)\rangle \quad \text{(from Eqn. 47)} \\
&= \frac{1}{2\tau} (Z * Z)(h) \quad \text{(from Eqn. 10)} \\
&= \frac{1}{2\sqrt{\tau}} Y(h) \quad \text{(from Eqn. 60)}
\end{aligned}$$

Doing a Fourier transform, the spectral power density is

$$\begin{aligned}
\rho_k(k) &= \frac{1}{\sqrt{\tau}} \tilde{r}_{EE}(k) \\
&= \frac{1}{2\tau} \tilde{Y}(k) \\
\rho_k(k) &= \frac{\hbar c\, |k|}{2\tau \varepsilon_0 S} \quad \text{(from Eqn. 60)}
\end{aligned}$$

This is another very interesting result. This says there is an infinite amount of power in the vacuum E field. Many authors, including Dougherty[1], say that stochastic processes of infinite power, like white noise or like this vacuum E field, are physically impossible, yet here is a counterexample to this assertion. On the surface, this result still seems physically impossible, but keep in mind that any quantum measurement requires interaction and may involve energy transfer. So when the E field is measured and found to be non-zero, the resulting energy did not need to have come from the vacuum. It could have easily come from the device that was doing the measuring.

MEASURING THE VACUUM E FIELD IN REAL LIFE

Riek et al. determined experimentally that "the variance is inversely proportional to the four-dimensional spacetime volume"[2]. This would imply the standard deviation should be inversely proportional to the square root of the window width, seemingly contradicting the theoretical result expressed with equation 57. The discrepancy can be explained though. Their method was to emit a femtosecond probe pulse of around one and a half cycles through an EOX electro-optic crystal to measure the vacuum E field comoving with the pulse. The vacuum E field altered the crystal's refractive index, changing the polarisation of the probe pulse. This change was then measured using ellipsometry. This amounts to an E field measurement with an averaging window with the same shape as the probe pulse. Something like this:

$$W(x) = \frac{1}{h}\cos(k_0 x) W_0(\tfrac{x}{h}) \tag{63}$$

The pulse's central wavenumber is k_0 and W_0 is the pulse's envelope. This turns equation 57 into

$$\sigma(\overline{E}_v) \approx \sqrt{\frac{\hbar c k_0}{4\varepsilon_0 S h}\int_u |\tilde{W}_0(u)|^2}. \tag{64}$$

When the central wavenumber and window width are treated separately, $\frac{1}{\sqrt{h}}$ dependence is recovered in the standard deviation. The approximation holds as long as the pulse's spectrum is relatively closely distributed around a central frequency, which has to be the case in order for the pulse to propagate. Even a pulse lasting only one and a half cycles will have a relatively localised distribution around the central frequency.

CONCLUSIONS, LIMITATIONS

Quantum mechanics is endlessly subtle and fascinating and vacuum EM field measurements are no exception. The infinite length single-polarisation dispersion-free waveguide is a very useful theoretical model to explore these subtleties. The same model can be used to explore the statistical properties of number states and coherent states. The model clearly has limitations, though. Extending to three dimensions would be trivial, except that doing so necessarily involves addressing the constraints on the field, $\nabla \cdot E = 0$ and $\nabla \cdot A = 0$. Though such details would have only served to obscure the purpose of this project, it would be interesting to see what these constraints do to the theory.

[1] E. R. Dougherty, *Random processes for image and signal processing (1st ed.)* (SPIE Optical Engineering Press, New York, 1999).

[2] C. Riek, D. V. Seletskiy, A. S. Moskalenko, J. F. Schmidt, P. Krauspe, S. Eckart, S. Eggert, G. Burkard, and A. Leitenstorfer, Science **350** (2015).